

Calculators, Mobile Phones, Pagers and all other mobile communication equipment are not allowed.

1. Let  $f(x) = \sqrt{x}$ .

a) Find  $dy$  at  $x = 9$  with  $\Delta x = -0.1$ .

b) Use differentials to approximate  $f(8.9)$ .

(4 Points)

2. Find an equation of the normal line to the graph of  $y = x^2 + x \sin y + \frac{\pi}{2}$  at  $x = 0$ .

(4 Points)

3. Let  $x$  and  $y$  be two real numbers where:  $x - y = 20$ . Find  $x$  and  $y$  such that  $P = x^2 + y^2$  is minimum.

(4 Points)

4. Let  $f(x) = \sqrt[3]{x^2}$ ,  $a = -1$ , and  $b = 8$ .

a) Show that there is no point  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

b) Explain why the result in part (a) does not contradict the Mean Value Theorem.

(4 Points)

5. Let  $f(x) = \frac{3x^2 - 1}{x^3}$ .

a) Find the vertical and horizontal asymptotes of  $f$  (if any).

b) Show that  $f'(x) = \frac{3(1 - x^2)}{x^4}$ . Find the intervals on which  $f$  is increasing and the intervals on which  $f$  is decreasing and then find the local extrema of  $f$  (if any).

c) Given that  $f''(x) = \frac{6(x^2 - 2)}{x^5}$ . Find the intervals on which the graph of  $f$  is concave upward and the intervals on which the graph of  $f$  is concave downward. Find the points of inflection (if any).

d) Discuss the symmetry of the graph of  $f$ .

(9 Points)

e) Sketch the graph of  $f$ .

$$f(x) = \sqrt{x} \quad , \quad f'(x) = \frac{1}{2\sqrt{x}} \quad ; \quad f(9) = 3 \quad ; \quad f'(9) = \frac{1}{6}$$

$$a) \quad dy = f'(x) dx = \frac{1}{6} (-0.1) = -\frac{0.1}{6}$$

$$b) \quad f(8.9) = f(9) + f'(9)(-0.1) = 3 - \frac{0.1}{6} = \frac{179}{60}$$

$$2) \quad y' = 2x + (1) \sin y + x y' \cos y \Rightarrow y'(1 - x \cos y) = 2x + \sin y$$

$$\Rightarrow y' = \frac{2x + \sin y}{1 - x \cos y}$$

when  $x=0, y = \pi/2 \Rightarrow \left. \frac{dy}{dx} \right|_{(0, \pi/2)} = 1 \Rightarrow m = -1, \boxed{y = -x + \pi/2}$ .

$$3) \quad x = 20 + y \quad (\text{or}) \quad y = x - 20$$

$$P = x^2 + (x - 20)^2 \Rightarrow \frac{dP}{dx} = 2x + 2(x - 20) = 0 \Rightarrow 4x = 40 \Rightarrow x = 10$$

$$\Rightarrow y = -10 \quad ; \quad P''(x) = 4$$

$$\left. \frac{d^2P}{dx^2} \right|_{x=10} = 4 > 0 \Rightarrow P(10) \text{ is minimum, } S = \{(10, -10)\}$$

$$4) \quad f(x) = \sqrt[3]{x^2} \quad , \quad f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

$$a) \quad f'(c) = \frac{2}{3\sqrt[3]{c}} = \frac{f(8) - f(-1)}{8 - (-1)} = \frac{4 - 1}{9} = \frac{1}{3} \Rightarrow \sqrt[3]{c} = 2$$

$$\Rightarrow c = 8 \notin (-1, 8).$$

b) It does not contradict M.V.T. because  $f$  is not differentiable @  $x=0 \in (-1, 8)$ . Actually,  $f$  is cont. @  $x=0$ ,

$\lim_{x \rightarrow 0} f'(x) = \pm \infty \Rightarrow$  The graph of  $f$  has a V.T. at  $x=0$ .

$$f(x) = \frac{3x^2 - 1}{x^3} \quad ; \quad D_f = \mathbb{R} \setminus \{0\} \quad ; \quad \begin{array}{l} \text{x-intercepts} \\ 3x^2 - 1 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \end{array}$$

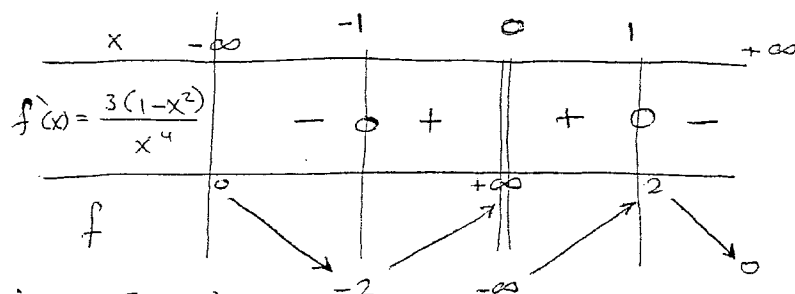
$$a) \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{3x^2 - 1}{x^3} = \frac{-1}{0^+} = -\infty \quad ; \quad \lim_{x \rightarrow 0^-} f(x) = \frac{-1}{0^-} = +\infty$$

$\Rightarrow x=0$  is a V.A.

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2(3 - \frac{1}{x^2})}{x^3} = \lim_{x \rightarrow -\infty} \frac{3}{x} = 0 \quad ; \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{3}{x} = 0$$

$\Rightarrow y=0$  is a H.A.

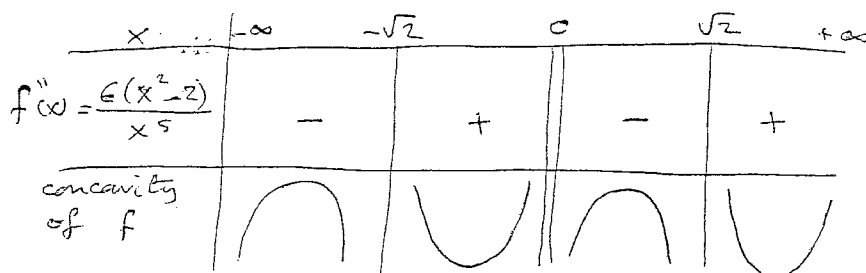
$$b) f'(x) = \frac{6x(x^3) - 3x^2(3x^2 - 1)}{x^6} = \frac{-3x^4 + 3x^2}{x^6} = \frac{3x^2(1 - x^2)}{x^6} = \frac{3(1 - x^2)}{x^4}$$



i)  $f$  is increasing on  $[-1, 0) \cup (0, 1]$  and decreasing on  $(-\infty, -1] \cup [1, +\infty)$

ii)  $f(-1) = -2$  is a local minima;  $f(1) = 2$  is a local maxima.

$$c) f''(x) = f''(x) = \frac{6(x^2 - 2)}{x^5}$$



i)  $f$  is concave upward on  $[-\sqrt{2}, 0) \cup [\sqrt{2}, +\infty)$  and downward on  $(-\infty, -\sqrt{2}] \cup (0, \sqrt{2}]$ .

ii)  $(-\sqrt{2}, f(-\sqrt{2}))$  and  $(\sqrt{2}, f(\sqrt{2}))$  are points of inflection.

$$d) f(-x) = \frac{3(-x)^2 - 1}{(-x)^3} = -\frac{3x^2 - 1}{x^3} = -f(x) \Rightarrow f \text{ is odd}$$

$\Rightarrow f$  is symmetric w.r.t. the origine.

e)

