kuwait University
Math. & Comp. Sci. Dept.

Math 101
Second Exam

July 31st, 2003
Time: 75 minutes

Calculators, Mobile Phones, Pagers and all other mobile communication equipment are not allowed.

1. Let $f(x) = \sqrt{x}$.

   a) Find dy at $x = 9$ with $\Delta x = -0.1$.

   b) Use differentials to approximate $f(8.9)$.

(4 Points)

2. Find an equation of the normal line to the graph of $y = x^2 + x\sin y + \frac{\pi}{2}$ at $x = 0$.

(4 Points)

3. Let x and y be two real numbers where: $x - y = 20$. Find x and y such that $P=x^2+y^2$ is minimum.

(4 Points)

4. Let $f(x) = \sqrt[3]{x^2}$, $a = -1$, and $b = 8$.

   a) Show that there is no point c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   b) Explain why the result in part (a) does not contradict the Mean Value Theorem.

(4 Points)

5. Let $f(x) = \dfrac{3x^2 - 1}{x^3}$.

   a) Find the vertical and horizontal asymptotes of f (if any).

   b) Show that $f'(x) = \dfrac{3(1 - x^2)}{x^4}$. Find the intervals on which f is increasing and the intervals on which f is decreasing and then find the local extrema of f (if any).

   c) Given that $f''(x) = \dfrac{6(x^2 - 2)}{x^5}$. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection (if any).

   d) Discuss the symmetry of the graph of f.

(9 Points)

   e) Sketch the graph of f.

1) $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$; $f(9)=3$; $f'(9)=\frac{1}{6}$

a) $dy=f'(x)\,dx=\frac{1}{6}(-0.1)=\frac{-0.1}{6}$

b) $f(8.9)=f(9)+f'(9)(-0.1)=3-\frac{0.1}{6}=\frac{179}{60}$

2) $y'=2x+(1)\sin y+xy'\cos y \Rightarrow y'(1-x\cos y)=2x+\sin y$
$\Rightarrow y'=\frac{2x+\sin y}{1-x\cos y}$

when $x=0$, $y=\pi/2 \Rightarrow \left.\frac{dy}{dx}\right|_{(0,\frac{\pi}{2})}=1 \Rightarrow m=-1$, $\boxed{y=-x+\pi/2}$.

3) $x=20+y$ (or) $y=x-20$

$P=x^2+(x-20)^2 \Rightarrow \frac{dP}{dx}=2x+2(x-20)=0 \Rightarrow 4x=40 \Rightarrow x=10$
$\Rightarrow y=-10$; $P''(x)=4$

$\left.\frac{d^2P}{dx^2}\right|_{x=10}=4>0 \Rightarrow P(10)$ is minimum , $S=\{(10,-10)\}$

4) $f(x)=\sqrt[3]{x^2}$, $f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\sqrt[3]{x}}$

a) $f'(c)=\frac{2}{3\sqrt[3]{c}}=\frac{f(8)-f(-1)}{8-(-1)}=\frac{4-1}{9}=\frac{1}{3} \Rightarrow \sqrt[3]{c}=2$
$\Rightarrow c=8\notin(-1,8)$.

b) It does not contradict M.V.T. because f is not differentiable @ $x=0\in(-1,8)$. Actually, f is conts. @ $x=0$, $\lim_{x\to 0^{\pm}}f'(x)=\pm\infty \Rightarrow$ The graph of f has a V.T. at $x=0$.

$f(x) = \dfrac{3x^2-1}{x^3}$; $D_f = \mathbb{R} \setminus \{0\}$,

<u>x – intercepts</u>

$3x^2 - 1 = 0 \Rightarrow x = \pm \dfrac{1}{\sqrt{3}}$.

a) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{3x^2-1}{x^3} = \dfrac{-1}{0^+} = -\infty$; $\lim\limits_{x \to 0^-} f(x) = \dfrac{-1}{0^-} = +\infty$

$\Rightarrow x = 0$ is a V.A .

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{x^2(3 - \frac{1}{x^2})}{x^3} = \lim\limits_{x \to -\infty} \dfrac{3}{x} = 0$; $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{3}{x} = 0$

$\Rightarrow y = 0$ is a H. A .

b) $f'(x) = \dfrac{6x\,(x^3) - 3x^2(3x^2-1)}{x^6} = \dfrac{-3x^4 + 3x^2}{x^6} = \dfrac{3x^2(1-x^2)}{x^6} = \dfrac{3(1-x^2)}{x^4}$.

x	$-\infty$		-1		0		1		$+\infty$
$f'(x) = \dfrac{3(1-x^2)}{x^4}$		$-$	0	$+$	$\Vert$	$+$	0	$-$	
f	0	$\searrow$	-2	$\nearrow$	$+\infty \Vert -\infty$	$\nearrow$	2	$\searrow$	0

i) f is increasing on $[-1, 0) \cup (0, 1]$ and decreasing on $(-\infty, -1] \cup [1, +\infty)$

ii) $f(-1) = -2$ is a local minima ; $f(1) = 2$ is a local maxima .

c) $f''(x) = f''(x) = \dfrac{6(x^2-2)}{x^5}$

x	$-\infty$		$-\sqrt{2}$		0		$\sqrt{2}$		$+\infty$
$f''(x) = \dfrac{6(x^2-2)}{x^5}$		$-$		$+$	$\Vert$	$-$		$+$	
concavity of f					$\Vert$				

i) f is concave upward on $[-\sqrt{2}, 0) \cup [\sqrt{2}, +\infty)$ and downward on $(-\infty, -\sqrt{2}] \cup (0, \sqrt{2}]$.

ii) $(-\sqrt{2}, f(-\sqrt{2}))$ and $(\sqrt{2}, f(\sqrt{2}))$ are points of inflection .

d) $f(-x) = \dfrac{3(-x)^2 - 1}{(-x)^3} = -\dfrac{3x^2-1}{x^3} = -f(x) \Rightarrow f$ is odd

$\Rightarrow f$ is symmetric w.r.t. the origine.

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